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Abelian para-Kähler structures on Lie algebras

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ABSTRACT

Para-Kähler Lie algebras which decompose as the sum of two abelian Lagrangian subalgebras are studied. We propose several constructions and provide an inductive description of such Lie algebras. The curvatures of the para-Kähler metric are computed and sufficient conditions to ensure flatness or Ricci-flatness are given. The Lie algebras for which the para-Kähler metric is Einstein and non-Ricci-flat are completely characterized.

1. Introduction

An almost paracomplex structure on a manifold M is given by an endomorphism K of its tangent bundle such that $K^2 = \mathrm{id}$ and the two eigendistributions $T^{\pm}M = \ker(K \mp \mathrm{id})$ have the same rank. The almost paracomplex structure is said to be integrable if the eigendistributions are involutive. As in the complex case, the integrability of an almost paracomplex structure is equivalent to the vanishing of the Nijenhuis tensor N_K of the almost paracomplex structure K, where $N_K(X,Y) = [X,Y] + [KX,KY] - K[KX,Y] - K[X,KY]$ for X, Y vector fields on M. If M admits a pseudo-Riemannian metric g and a skew-symmetric paracomplex structure K which is parallel with respect to the Levi-Civita connection, then the triple (M, g, K) is said to be a para-Kähler manifold. It is well known that this is equivalent to the existence of a symplectic structure $\omega = g \circ (\mathrm{id} \times K)$ on M admitting two complementary involutive Lagrangian distributions [1].

When the manifold is a Lie group G and the metric and the paracomplex structure are considered left-invariant, they are both determined by their restrictions to the Lie algebra $\mathfrak{g}$ of G. Further, the existence of a left invariant para-Kähler structure on G reduces to the existence of a symplectic form on the Lie algebra $\mathfrak{g}$ admitting two complementary Lagrangian subalgebras. In such situation, we will say that we have a para-Kähler structure on the Lie algebra $\mathfrak{g}$. A well-known example of such para-Kähler structures is provided by algebras admitting Manin pairs [4] in which the para-Kähler metric is actually bi-invariant. Such algebras must be necessarily nilpotent. It should be noticed, however, that if the para-Kähler metric is bi-invariant then the Lagrangian subalgebras may not be abelian unless $\mathfrak{g}$ is itself abelian.

In this paper we will consider a special class of left-invariant para-Kähler structures. Namely, we will consider those for which the Lagrangian subalgebras $\mathfrak{g}^{\pm}$ are abelian subalgebras. One says in such case that the paracomplex structure is

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abelian and it can be seen that also non-nilpotent solvable Lie algebras may appear. Our main interest is to give a description of such algebras and study the properties of their curvature.

The paper is organized as follows. In Section 2 we recall some basic definitions and elementary results on para-Kähler structures on Lie algebras. We see that the class of para-Kähler Lie algebras for abelian paracomplex structures is a wide one in Section 3 and propose several constructions of such Lie algebras. In Sections 4 and 5 we show that the eigenspaces of K are actually associative commutative algebras, we study the main structural properties of para-Kähler Lie algebras with abelian paracomplex structure and give an inductive description of such algebras by means of the so called symplectic double extension method [8,12]. Finally, Section 6 contains all the results on the curvature of the para-Kähler metric. We give some sufficient conditions to ensure that the metric is flat or Ricci-flat and we completely describe the non-Ricci-flat cases in which the metric is Einstein.

2. Preliminaries

We first recall some basic definitions [1–4,10,11]. All the algebras in the paper are considered finite-dimensional over a field $\mathbb{K}$ of characteristic 0.

Definition 2.1. A *product structure* on a Lie algebra $\mathfrak{g}$ is a linear map $K \in \mathfrak{gl}(\mathfrak{g})$ such that $K^2 = \mathrm{id}_{\mathfrak{g}}$, $K \neq \mathrm{id}_{\mathfrak{g}}$ and

$$[x, y] = K[Kx, y] + K[x, Ky] - [Kx, Ky] \quad \text{for all } x, y \in \mathfrak{g}.$$

The product structure K provides a decomposition of the vector space $\mathfrak{g}$ as the direct sum of the eigenspaces $\mathfrak{g}_+ = \ker(K - \mathrm{id}_{\mathfrak{g}})$, $\mathfrak{g}_- = \ker(K + \mathrm{id}_{\mathfrak{g}})$. Actually, $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are subalgebras of $\mathfrak{g}$. When both subalgebras $\mathfrak{g}_\pm$ have the same dimension, the product structure is said to be a *paracomplex structure* on $\mathfrak{g}$.

Two Lie algebras endowed with paracomplex structures $(\mathfrak{g}_1, K_1)$, $(\mathfrak{g}_2, K_2)$ are said to be *paraholomorphically equivalent* if there is an isomorphism $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $K_1 = \psi^{-1} \circ K_2 \circ \psi$.

We will say that a product structure K is *abelian* if $[Kx, Ky] = -[x, y]$ holds for all $x, y \in \mathfrak{g}$. In such case one easily verifies that the subalgebras $\mathfrak{g}_\pm$ are actually abelian.

Definition 2.2. We say that a Lie algebra $\mathfrak{g}$ admits a *symplectic structure* if it admits a non-degenerate scalar 2-cocycle ω. The pair $(\mathfrak{g}, \omega)$ is said to be a *symplectic Lie algebra*.

Remark 2.1. It is well known that a symplectic Lie algebra $(\mathfrak{g}, \omega)$ is naturally endowed with a structure of left-symmetric algebra compatible with the Lie structure. Explicitly, the product defined by $\omega(x.y, z) = -\omega(y, [x, z])$ for $x, y, z \in \mathfrak{g}$ verifies $[x, y] = x.y - y.x$ and

$$[x, y].z = x.(y.z) - y.(x.z) \quad \text{for all } x, y, z \in \mathfrak{g}.$$

Obviously, this shows that each Lie group with Lie algebra $\mathfrak{g}$ can be equipped with the torsion-free flat invariant connection defined by $\nabla^\omega_x y = x.y$ for $x, y \in \mathfrak{g}$.

Definition 2.3. Let $(\mathfrak{g}, \omega)$ be a symplectic Lie algebra and K a paracomplex structure on $\mathfrak{g}$ such that $\omega(Kx, Ky) = -\omega(x, y)$ for all $x, y \in \mathfrak{g}$. The triple $(\mathfrak{g}, \omega, K)$ will be called a *para-Kähler Lie algebra*. If the paracomplex structure is abelian we will say that the algebra $\mathfrak{g}$ is endowed with an *abelian para-Kähler structure*.

Remark 2.2. The existence of an abelian para-Kähler structure on a symplectic Lie algebra $(\mathfrak{g}, \omega)$ is equivalent to the fact that $\mathfrak{g}$ admits a vector space decomposition $\mathfrak{g} = W_1 \oplus W_2$ where each W_i is a completely isotropic abelian subalgebra. Note than, in such case, the maps $K = \pm(\mathrm{pr}_1 - \mathrm{pr}_2)$, where pr_i denotes the corresponding projection on W_i, are abelian paracomplex structures which obviously verify the compatibility condition $\omega(Kx, Ky) = -\omega(x, y)$ for $x, y \in \mathfrak{g}$.

In a Lie algebra with abelian para-Kähler structure, the paracomplex structure commutes with right multiplications of the left-symmetric product induced by ω. Explicitly,

Lemma 2.1. *Let $(\mathfrak{g}, \omega, K)$ be a para-Kähler Lie algebra where K is abelian. The left-symmetric product defined by ω on $\mathfrak{g}$ verifies $K(x.y) = (Kx).y$ for all $x, y \in \mathfrak{g}$.*

Proof. If $x, y, z \in \mathfrak{g}$ then we have

$$\omega\big(K(x.y), z\big) = -\omega(x.y, Kz) = \omega\big(y, [x, Kz]\big) = -\omega\big(y, [Kx, z]\big) = \omega\big((Kx).y, z\big),$$

which proves the result. $\square$

The proof of the following result can be explicitly found in [2]. It obviously shows that every para-Kähler Lie algebra with abelian paracomplex structure must be 2-step solvable.

Lemma 2.2. *If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{g}_+, \mathfrak{g}_-$ two abelian Lie subalgebras such that, as vector spaces, $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ then $\mathfrak{g}$ is 2-step solvable.*

The following definitions will be used several times through the paper (see, for instance, [7,4]):

Definition 2.4. A finite-dimensional associative algebra $\mathcal{U}$ (not necessarily unital) is said to be a *Frobenius algebra* if $\mathcal{U}$ is equipped with a non-degenerate bilinear form, called the *Frobenius form*, $B : \mathcal{U} \times \mathcal{U} \to \mathbb{K}$ that satisfies $B(xy, z) = B(x, yz)$ for $x, y, z \in \mathcal{U}$.

A Frobenius algebra $(\mathcal{U}, B)$ is called a *symmetric algebra* if B is symmetric.

Definition 2.5. A Lie algebra $\mathfrak{g}$ is said to be a *quadratic Lie algebra* if it admits a non-degenerate symmetric bilinear form B such that $B([x, y], z) = B(x, [y, z])$ holds for all $x, y, z \in \mathfrak{g}$.

3. Examples of para-Kähler Lie algebras with abelian paracomplex structure

As we will see, the family of Lie algebras admitting abelian para-Kähler structures is a wide one. In this section we propose several constructions of such Lie algebras.

3.1. The 2-dimensional Lie algebras

If $\mathfrak{g}$ is a Lie algebra of dimension 2 then obviously every skew-symmetric non-degenerate bilinear form ω provides a symplectic structure. Take an arbitrary basis $\{x, y\}$ in $\mathfrak{g}$ and define $Kx = x$, $Ky = -y$. It is straightforward to prove that K is abelian and $(\mathfrak{g}, \omega, K)$ is para-Kähler.

3.2. The Lie algebra $\mathfrak{aff}(\mathcal{A})$ of a Frobenius commutative algebra

Let $\mathcal{A}$ be a commutative associative algebra. The vector space $\mathcal{A} \oplus \mathcal{A}$ with the product defined by $[(a, b), (a', b')] := (0, ab' - a'b)$ for all $a, a', b, b' \in \mathcal{A}$, is a Lie algebra denoted by $\mathfrak{aff}(\mathcal{A})$ [5]. Note that, actually, $\mathfrak{aff}(\mathcal{A}) = \mathfrak{aff}(\mathbb{R}) \otimes \mathcal{A}$ with the bracket $[x \otimes a, y \otimes a'] = [x, y] \otimes aa'$. It is clear that the linear map K on $\mathfrak{aff}(\mathcal{A})$ defined by: $K(a, b) := (b, a)$, for $a, b \in \mathcal{A}$, is an abelian product structure on $\mathfrak{aff}(\mathcal{A})$. If, in addition, we suppose that B is a Frobenius form, then the bilinear form ω of $\mathfrak{aff}(\mathcal{A})$ defined by

$$\omega\big((a, b), \big(a', b'\big)\big) = \frac{1}{2}\big(B\big(a, a'\big) - B\big(a', a\big) - B\big(b, b'\big) + B\big(b', b\big)\big) + \frac{1}{2}\big(B\big(a, b'\big) + B\big(b', a\big) - B\big(b, a'\big) - B\big(a', b\big)\big)$$

for $a, a', b, b' \in \mathcal{A}$, defines a symplectic structure on Lie algebra $\mathfrak{aff}(\mathcal{A})$ and a simple computation shows that $(\mathfrak{aff}(\mathcal{A}), \omega, K)$ is para-Kähler.

The special case in which the pair $(\mathcal{A}, B)$ is actually a symmetric algebra will have a special relevance in the sequel. In such case, we will say that the para-Kähler structure given above is the *standard abelian para-Kähler structure on* $\mathfrak{aff}(\mathcal{A})$. Notice that, then, the symplectic form is simply given by

$$\omega\big((a, b), \big(a', b'\big)\big) := B\big(a, b'\big) - B\big(b, a'\big), \quad a, a', b, b' \in \mathcal{A}.$$

We recall that starting from an arbitrary associative commutative algebra one may construct a symmetric algebra by the method of T*-extension described in [6].

3.3. Construction of para-Kähler Lie algebras from a couple of quadratic Lie algebras

Let $(\mathcal{L}, [-,-], B)$ be a quadratic 2-nilpotent Lie algebra. On the vector space $\mathcal{A} := \mathcal{L} \times \mathcal{L}$ we consider the commutative product defined by:

$$\big(a, a'\big) \cdot \big(b, b'\big) = \big(\big[a', b\big] + \big[b', a\big], \big[a, b'\big] + \big[b, a'\big]\big), \quad \text{for all } a, a', b, b' \in \mathcal{L}.$$

Since $\mathcal{L}$ is 2-nilpotent, it is obvious that $\mathcal{A}^3 = \{0\}$ and therefore $(\mathcal{A}, \cdot)$ is an associative commutative algebra.

If we consider the map $\mathcal{K} : \mathcal{A} \to \mathcal{A}$ defined by $\mathcal{K}(a, a') = (a, -a')$, one can easily verify that $\mathcal{K}(a, a') \cdot \mathcal{K}(b, b') = -(a, a') \cdot (b, b')$. Further, the non-degenerate skew-symmetric bilinear form Ω defined on $\mathcal{A}$ by: $\Omega((0, l), (l', 0)) = B(l, l')$ for all $l, l' \in \mathcal{L}$ verifies that

$$\sum_{(1,2,3)}^{cyc} \Omega\big(\big(a_1, a_1'\big) \cdot \big(a_2, a_2'\big), \big(a_3, a_3'\big)\big) = 0, \qquad \Omega\big(\mathcal{K}\big(a_1, a_1'\big), \big(a_2, a_2'\big)\big) = -\Omega\big(\big(a_1, a_1'\big), \mathcal{K}\big(a_2, a_2'\big)\big)$$

holds for all $a_i, a_i' \in \mathcal{L}$ $(1 \leqslant i \leqslant 3)$, where the symbol $\sum_{(1,2,3)}^{cyc}$ stands for cyclic sum.

Thus, we have a triple $(\mathcal{A}, \Omega, \mathcal{K})$ where $\mathcal{A}$ is an associative commutative algebra, Ω a non-degenerate scalar 2-cocycle for the Hochschild cohomology on $\mathcal{A}$ and $\mathcal{K} \in \mathfrak{gl}(\mathcal{A})$ such that

$$\mathcal{K}^2 = \mathrm{id}_{\mathcal{A}}, \qquad \mathcal{K}(s).\mathcal{K}(t) = -s.t, \qquad \Omega\big(\mathcal{K}(s), t\big) = -\Omega\big(s, \mathcal{K}(t)\big), \quad \forall s, t \in \mathcal{A}. \tag{1}$$

Now, let $(\mathfrak{g}, [-,-]_{\mathfrak{g}}, B')$ be an arbitrary quadratic Lie algebra. The vector space $\mathfrak{g} \otimes \mathcal{A}$ with the bracket $[x \otimes s, y \otimes t] := [x, y]_{\mathfrak{g}} \otimes s.t$ for $(x, s), (y, t) \in \mathfrak{g} \times \mathcal{A}$, is a Lie algebra. The bilinear form ω defined on $\mathfrak{g} \otimes \mathcal{A}$ by $\omega(x \otimes s, y \otimes t) := B'(x, y)\Omega(s, t)$ is a symplectic structure on the Lie algebra $(\mathfrak{g} \otimes \mathcal{A}, [-,-]_{\mathfrak{g}})$. Finally, if we define the map $K := \mathrm{id}_{\mathfrak{g}} \otimes \mathcal{K}$, then we have that (ω, K) provides an abelian para-Kähler structure on $\mathfrak{g} \otimes \mathcal{A}$.

Remark 3.1. It should be noticed that, regardless of the definition of $\mathcal{A}$ above, whenever a triple $(\mathcal{A}, \Omega, \mathcal{K})$, where $\mathcal{A}$ is an associative commutative algebra, Ω a non-degenerate scalar 2-cocycle for the Hochschild cohomology on $\mathcal{A}$ and $\mathcal{K} \in \mathfrak{gl}(\mathcal{A})$, verifies the conditions (1) then the construction of an abelian para-Kähler structure on $\mathfrak{g} \otimes \mathcal{A}$ for a quadratic $\mathfrak{g}$ is still valid.

It can be shown that an associative algebra $\mathcal{A}$ in which (1) holds must be nilpotent (in fact, $\mathcal{A}^4 = \{0\}$) since the para-Kähler Lie algebra $\mathfrak{sl}(2) \otimes \mathcal{A}$ must be 2-solvable and $\mathfrak{sl}(2)$ is a perfect Lie algebra.

4. Decomposition into a sum of associative commutative algebras

Lemma 4.1. *Let us consider an abelian para-Kähler structure (ω, K) on a Lie algebra $\mathfrak{g}$ and denote $U = \ker(K - \mathrm{id}_{\mathfrak{g}})$, $V = \ker(K + \mathrm{id}_{\mathfrak{g}})$.*

The abelian Lie subalgebras U, V are left-symmetric subalgebras of $\mathfrak{g}$ endowed with the left-symmetric product defined by the symplectic form ω. Hence they are, actually, commutative associative algebras.

Proof. For every $u, u', u'' \in U$ one has $\omega(uu', u'') = -\omega(u', [u, u'']) = 0$ and, hence $uu' \in U^{\perp} = U$, which shows that U is a subalgebra for the left-symmetric product. An analogous reasoning proves the result for V. The fact that a commutative left-symmetric algebra must be associative completes the proof. $\square$

Remark 4.1. It is obvious that the vector spaces U^* and V are isomorphic under the isomorphism $\varphi : V \to U^*$ given by $\varphi(v) = \omega(v, -)$ for all $v \in V$. We can, therefore, define an associative commutative product $*$ on U^* by defining

$$\omega(v, -) * \omega\big(v', -\big) = \omega\big(vv', -\big), \quad v, v' \in V.$$

Under such identification, we may consider $\mathfrak{g} = U \oplus U^*$ endowed with the symplectic form ω given by

$$\omega\big(u + f, u' + f'\big) = f\big(u'\big) - f'(u), \quad u, u' \in U, \quad f, f' \in U^*.$$

Proposition 4.2. *Let $(\mathfrak{g}, \omega, K)$ be a para-Kähler Lie algebra with abelian paracomplex structure and consider $U = \ker(K - \mathrm{id}_{\mathfrak{g}})$ endowed with the associative product defined by ω and $(U^*, *)$ as above. Let us denote by R_u, the multiplication by an element $u \in U$ on U.*

The left-symmetric product defined by ω on the vectorial sum $\mathfrak{g} = U \oplus U^*$ is completely defined by its restrictions to U and U^*. Actually, for each $u \in U$, $f \in U^*$, one has $f.u' = f \circ R_{u'} \in U^*$ and $u.f \in U$ is defined as the unique element in U such that $f'(u.f) = f' * f(u)$ for all $f' \in U^*$.

Proof. Let us consider $u, u' \in U$, $f, f' \in U^*$. We then have

$$\omega\big(f.u, u' + f'\big) = -\omega\big(u, \big[f, u'\big]\big) = -\omega\big(u'u, f\big) = f\big(u'u\big) = (f \circ R_u)\big(u'\big) = \omega\big(f \circ R_u, u' + f'\big),$$

showing that $f.u' = f \circ R_{u'}$. Similarly,

$$\omega\big(u.f, u'\big) = -\omega\big(f, \big[u, u'\big]\big) = 0,$$
$$f'(u.f) = \omega\big(f', u.f\big) = -\omega\big(\big[u, f'\big], f\big) = \omega\big(\big[f', u\big], f\big) = -\omega\big(u, f' * f\big) = f' * f(u)$$

which proves that $u.f \in U$ and verifies the stated identity. $\square$

Theorem 4.3. *Let $(U, \cdot)$ be a commutative associative algebra and suppose that we have a commutative associative law $*$ in U^*. Let us define on the vector space $\mathfrak{g} = U \oplus U^*$ the product defined by $u.u' = u \cdot u'$, $f.f' = f * f'$, $f.u = f \circ R_u$, where R_u denotes the multiplication by u in U, and $u.f \in U$ is the unique element in U such that $g(u.f) = g * f(u)$ for all $g \in U^*$.*

Then $\mathfrak{g}$ endowed with such product is a left-symmetric algebra if and only if for all $u, u' \in U$, $f, f' \in U^*$ one has:

$$\big(f * \big(f' \circ R_u\big)\big)\big(u'\big) = \big((f \circ R_u) * f'\big)\big(u'\big) = \big(f * \big(f' \circ R_{u'}\big)\big)(u).$$

Moreover, in such case, the bilinear form ω given by $\omega(u + f, u' + f') = f(u') - f'(u)$ for $u, u' \in U$, $f, f' \in U^*$ is a symplectic form on the Lie algebra $(\mathfrak{g}, [.,.])$ where $[x, y] = x.y - y.x$ and $(\mathfrak{g} = U \oplus U^*, \omega)$ is a para-Kähler Lie algebra with abelian paracomplex structure.

Proof. Let us consider $u, u' \in U$ and $f, f', g \in U^*$ then we have

$$f'\big([u,u'].f - u.(u'.f) + u'.(u.f)\big) = -\big(f' \circ R_u\big) * f\big(u'\big) + \big(f' \circ R_{u'}\big) * f(u),$$
$$\big[f, f'\big].u - f.\big(f'.u\big) + f'.(f.u) = -f * \big(f' \circ R_u\big) + f' * (f \circ R_u),$$

which shows the necessity of the identities. Further, if such identities hold then

$$g\big((u.f).u'\big) = g \circ R_{u'}(u.f) = (g \circ R_{u'}) * f(u) = g * (f \circ R_{u'})(u) = g\big(u.(f.u'\big),$$
$$g\big((u.f).f'\big) = \big(g * f'\big)(u.f) = \big(g * f'\big) * f(u) = g * \big(f * f'\big)(u) = g\big(u.(f.f'\big),$$
$$f.\big(u.f'\big)\big(u'\big) = f \circ R_{u.f'}\big(u'\big) = f \circ R_{u'}\big(u.f'\big) = f' * \big(f \circ R'_u\big)(u) = f' * (f \circ R_u)\big(u'\big)$$

and, therefore,

$$[u, f].f' - u.\big(f.f'\big) + f.\big(u.f'\big) = -(f \circ R_u) * f' + f' * (f \circ R_u) = 0,$$
$$[u, f].u' - u.\big(f.u'\big) + f.\big(u.u'\big) = (u.f).u' - (f \circ R_u) \circ R_{u'} - u.\big(f.u'\big) + f \circ R_{u \cdot u'} = 0.$$

Thus, if the identities hold, then $(\mathfrak{g}, .)$ is left-symmetric.

The skew-symmetric form given by $\omega(u + f, u' + f') = f(u') - f'(u)$ for $u, u' \in U$, $f, f' \in U^*$ is obviously non-degenerate and $\omega(U, U) = \omega(U^*, U^*) = \{0\}$. Hence, it only remains to show that ω is a 2-cocycle. For all $u_1, u_2, u_3 \in U$, $f_1, f_2, f_3 \in U^*$ one has,

$$\begin{aligned}\sum_{(1,2,3)}^{cyc} \omega\big([u_1 + f_1, u_2 + f_2], u_3 + f_3\big) &= \sum_{(1,2,3)}^{cyc} \omega(u_1.f_2 - f_2.u_1 + f_1.u_2 - u_2.f_1, u_3 + f_3)\\ &= \sum_{(1,2,3)}^{cyc} \big(-f_3(u_1.f_2) + f_3(u_2.f_1) - f_2.u_1(u_3) + f_1.u_2(u_3)\big)\\ &= \sum_{(1,2,3)}^{cyc} \big(-f_2 * f_3(u_1) + f_1 * f_3(u_2) - f_2(u_3u_1) + f_1(u_3u_2)\big),\end{aligned}$$

which obviously vanishes since the products on U and U^* are commutative. □

Remark 4.2. Obviously, the left-symmetric algebra defined above is not commutative unless both products on U and U^* are null. In general it is not either associative. Actually a necessary and sufficient condition to be associative reads $(f * f') \circ R_u = f * (f' \circ R_u)$ for all $u \in U$ and $f, f' \in U^*$.

It should be noticed that if either U or U^* is unitary then the resulting algebra $\mathfrak{g}$ is always associative. For instance, if $e \in U$ is a unity for the product in U then for all $u, u' \in U$ and $f, f' \in U^*$ we have

$$\begin{aligned}\big(f * f'\big) \circ R_u\big(u'\big) &= \big(f * f'\big)\big(uu'\big) = f * \big(f' \circ R_e\big)\big(uu'\big) = f * \big(f' \circ R_{uu'}\big)(e)\\ &= f * \big(f' \circ R_u \circ R_{u'}\big)(e) = f * \big(f' \circ R_u \circ R_e\big)\big(u'\big) = f * \big(f' \circ R_u\big)\big(u'\big).\end{aligned}$$

Examples 4.1.

1. *The left-symmetric product of the standard structure on $\mathfrak{aff}(\mathcal{A})$ is associative.*
   Let $(\mathcal{A}, B)$ be a commutative associative symmetric algebra. For $a, a', b, b', c, d' \in \mathcal{A}$ one has

$$\begin{aligned}\omega\big((a,b).\big(a',b'\big),(c,d)\big) &= -\omega\big(\big(a',b'\big),\big[(a,b),(c,d)\big]\big) = -\omega\big(\big(a',b'\big),(0, ad - bc)\big)\\ &= B\big(a', bc\big) - B\big(a', ad\big) = B\big(a'b, c\big) - B\big(a'a, d\big) = \omega\big(\big(-aa', -ba'\big),(c,d)\big).\end{aligned}$$

   Hence, $(a,b)(a',b') = (-aa', -ba')$ and a straightforward calculation shows that such product is associative.
2. *Constructions from an arbitrary associative commutative algebra.*
   If U is an associative commutative algebra and on U^* one considers the null product, the necessary and sufficient condition for the associativity of $\mathfrak{g} = U \oplus U^*$ given in Remark 4.2 is obviously verified. Therefore, one can always construct a para-Kähler Lie algebra with abelian paracomplex structure starting from an arbitrary associative and commutative algebra.
3. *A non-associative example.*
   Let us consider U the linear $\mathbb{K}$-span of $\{e_1, e_2, e_3, e_4\}$ with the associative commutative product defined by the non-trivial products $e_1^2 = e_2$, $e_1e_2 = e_3$ and on U^* let us define the associative commutative product given in which the only non-zero product for the dual basis $\{e_1^*, e_2^*, e_3^*, e_4^*\}$ of $\{e_1, e_2, e_3, e_4\}$ is $e_3^* * e_4^* = -(e_1^* + e_2^*)$. It can be easily proved that

such products verify the compatibility conditions given in Theorem 4.3 and hence the vector space $\mathfrak{g} = U \oplus U^*$ admits an abelian para-Kähler structure. However, the left-symmetric product defined by the symplectic form is not associative since one has $(e_4^*.e_3^*).e_1 = -e_1^*$ but $e_4^*.(e_3^*.e_1) = e_4^*.e_2^* = 0$.

Proposition 4.4. *Let $(\mathfrak{g} = U \oplus U^*, \omega)$ be a para-Kähler Lie algebra with an abelian paracomplex structure. The left-symmetric product defined by ω on $\mathfrak{g}$ verifies for every $u, u', u'' \in U$, $f, f', f'' \in U^*$ the following identities*:

(i) $u.(u'.u'') = (u.u').u''$, $f.(f'.f'') = (f.f').f''$,
(ii) $f.(u.u') = (f.u).u'$, $u.(f.f') = (u.f).f'$,
(iii) $u.(f.u') = (u.f).u' = (u'.f).u$,
(iv) $f.(u.f') = (f.u).f' = (f'.u).f = f'.(u.f)$.

Proof. Since the restrictions of the left-symmetric product to U and U^* are associative, (i) is obvious. The first identity of (ii) is immediately deduced from $R_u \circ R_{u'} = R_{uu'}$. In order to prove (iii), let us consider $g \in U^*$. We then have

$$g\big(u.\big(f.u'\big)\big) = g * (f \circ R_{u'})(u) = f * (g \circ R_{u'})(u) = g \circ R_{u'}(u.f) = g\big((u.f).u'\big)$$

and also

$$g\big(u.\big(f.u'\big)\big) = g * (f \circ R_u)\big(u'\big) = f * (g \circ R_u)\big(u'\big) = g \circ R_u\big(u'.f\big) = g\big(\big(u'.f\big).u\big).$$

The second identity of (ii) as well as (iv) can be obtained by interchanging the roles of U and U^*. $\square$

Corollary 4.5. *Let $(\mathfrak{g} = U \oplus U^*, \omega)$ be a para-Kähler Lie algebra with abelian paracomplex structure. The following holds*:

(i) $\omega([\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]) = \{0\}$.
(ii) $\mathfrak{g}.[\mathfrak{g}, \mathfrak{g}] = \{0\}$,
(iii) $\mathrm{ann}(U)$ *and* $\mathrm{ann}(U^*)$ *are ideals of the Lie algebra* $\mathfrak{g}$, *where* $\mathrm{ann}(W)$ *stands for the annihilator of an associative commutative algebra W, this is to say,*

$$\mathrm{ann}(W) = \big\{w \in W:\ w.w' = 0 \text{ for all } w' \in W\big\}.$$

Proof. The equality $\omega([\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]) = \{0\}$ is deduced from (iii) in the proposition above since

$$u'.[u, f] = u'.(u.f) - u'.(f.u) = (u.f).u' - u'.(f.u) = \big(u'.f\big).u - u'.(f.u) = 0$$

implies $\omega([u, f], [u', f']) = -\omega(u'.[u, f], f') = 0$. The proof of (ii) now follows at once since $\omega(x.[y, z], t) = -\omega([y, z], [x, t]) = 0$. In order to prove that $\mathrm{ann}(U)$ is a Lie ideal of $\mathfrak{g}$ let us take $u_0 \in \mathrm{ann}(U)$. For every $f \in U^*$ one has $f.u_0 = f \circ R_{u_0} = 0$ and, therefore, $[u_0, u + f] = u_0.f \in U$ for every $u \in U$, $f \in U^*$. Further, by (iii) in the proposition above, $(u_0.f).u = (u.f).u_0 = 0$, which shows that $[u_0, u + f] \in \mathrm{ann}(U)$. The result for $\mathrm{ann}(V)$ is obtained in a similar way. $\square$

5. Inductive description of Lie algebras with abelian para-Kähler structures

In [12,8] the definitions of classical and generalized symplectic double extensions of a symplectic Lie algebra were given. In the following we will use some particular cases of such kind of extensions.

Definition 5.1. Let $(\mathfrak{g}, [-,-]_{\mathfrak{g}}, \omega)$ be a symplectic Lie algebra and let $D \in \mathrm{Der}(\mathfrak{g})$, $\lambda \in \mathbb{K}$ be such that the 2-cocycle

$$\Omega(x, y) = \omega\big(\big(\big(D + D^*\big)D + D^*\big(D + D^*\big) + \lambda\big(D + D^*\big)\big)x, y\big), \quad x, y \in \mathfrak{g}$$

is a 2-coboundary, this is to say, there exists $z \in \mathfrak{g}$ verifying $\Omega(x, y) = \omega([x, y], z)$ for all $x, y \in \mathfrak{g}$.

In the vector space $\tilde{\mathfrak{g}} = \mathbb{K}e \oplus \mathfrak{g} \oplus \mathbb{K}d$ let us define the bracket given by

$$[e, x] = 0, \qquad [e, d] = \lambda e, \qquad [x, y] = [x, y]_{\mathfrak{g}} + \omega\big(\big(D + D^*\big)x, y\big)e, \qquad [d, x] = D(x) + \omega(x, z)e$$

for all $x, y \in \mathfrak{g}$ and the skew-symmetric bilinear form $\tilde{\omega}$ defined by

$$\tilde{\omega}(d, e) = 1, \qquad \tilde{\omega}(x, e) = \tilde{\omega}(x, d) = 0, \qquad \tilde{\omega}(x, y) = \omega(x, y), \quad \forall x, y \in \mathfrak{g}.$$

The triple $(\tilde{\mathfrak{g}}, \tilde{\omega})$ is a symplectic Lie algebra called the *symplectic double extension of* $\mathfrak{g}$ *by a line defined by means of* D, λ, z. The double extension is said to be *classical* if $\lambda = 0$ and, hence, e is a central element in $\tilde{\mathfrak{g}}$.

The following result characterizes symplectic Lie algebras which are obtained by symplectic double extension by a line.

Proposition 5.1. *A symplectic Lie algebra $(\tilde{\mathfrak{g}}, \tilde{\omega})$ is a symplectic double extension of a symplectic Lie algebra by a line if and only if there exists $e \in [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]^\perp$ such that $\mathbb{K}e$ is an ideal of $\tilde{\mathfrak{g}}$.*

Proof. If $(\tilde{\mathfrak{g}}, \tilde{\omega})$ is a symplectic double extension of a symplectic Lie algebra $\mathfrak{g}$ by a line, it is obvious from the definition above that $e \in [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]^\perp$ and $\mathfrak{I} = \mathbb{K}e$ is an ideal of $\tilde{\mathfrak{g}}$. Conversely, suppose that in the symplectic Lie algebra $(\tilde{\mathfrak{g}}, \tilde{\omega})$ there exists an element with such two properties. Since $\tilde{\omega}$ is non-degenerate, one may find an element $d \in \tilde{\mathfrak{g}}$ such that $\omega(d, e) = 1$. Since $e \in [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]^\perp$ then $\mathfrak{I}^\perp$ is an ideal of $\tilde{\mathfrak{g}}$. The vector space $\mathfrak{g} = \mathfrak{I}^\perp \cap (\mathbb{K}d)^\perp$ admits the non-degenerate skew-symmetric form $\omega = \tilde{\omega}_{\mathfrak{g}\times\mathfrak{g}}$ and can be endowed with the Lie bracket $[-,-]_{\mathfrak{g}}$ defined by the projection of the bracket $[-,-]$ of $\tilde{\mathfrak{g}}$ on the subspace $\mathfrak{g}$. Actually, $(\mathfrak{g}, [-,-]_{\mathfrak{g}})$ is isomorphic to the Lie algebra $\mathfrak{I}^\perp/\mathfrak{I}$ and, moreover, $(\mathfrak{g}, [-,-]_{\mathfrak{g}}, \omega)$ happens to be a symplectic Lie algebra. Clearly, $\tilde{\mathfrak{g}} = \mathbb{K}e \oplus \mathfrak{g} \oplus \mathbb{K}d$ and $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] \subset \mathfrak{I}^\perp = \mathbb{K}e \oplus \mathfrak{g}$. The ideal $\mathfrak{I}$ must be central in $\mathfrak{I}^\perp$ because $\mathfrak{I}^\perp$ is an ideal of $\tilde{\mathfrak{g}}$ and $\tilde{\omega}$ is a symplectic form. Therefore, for every $x, y \in \mathfrak{g}$ one has:

$$[e, x] = 0, \qquad [e, d] = \lambda e, \qquad [d, x] = D(x) + \psi(x)e, \qquad [x, y] = [x, y]_{\mathfrak{g}} + \varphi(x, y)e$$

where $D \in \mathfrak{gl}(\mathfrak{g})$, $\psi \in \mathfrak{g}^*$ and $\varphi \in \mathfrak{g}^* \wedge \mathfrak{g}^*$. The condition $\tilde{\omega}([x, y], d) + \tilde{\omega}([y, d], x) + \tilde{\omega}([d, x], y) = 0$ is equivalent to $\varphi(x, y) = \omega((D + D^*)x, y)$ for all $x, y \in \mathfrak{g}$. Further, using Jacobi identity for $[d, [x, y]]$ one gets that D must be a derivation of $\mathfrak{g}$ and that

$$\psi\big([x, y]_{\mathfrak{g}}\big) = \omega\big(\big(\big(D + D^*\big)D + D^*\big(D + D^*\big) + \lambda\big(D + D^*\big)\big)x, y\big)$$

holds for all $x, y \in \mathfrak{g}$. Now, the result follows from the fact that, being ω non-degenerate, there exists $z \in \mathfrak{g}$ such that $\psi = \omega(-, z)$. □

Corollary 5.2. *Let $(\mathfrak{g}, \omega)$ be a solvable symplectic Lie algebra and suppose that one of the following conditions hold*:

(i) $\mathfrak{z}(\mathfrak{g}) \neq \{0\}$,
(ii) *$\mathbb{K}$ is algebraically closed and $C_{\mathfrak{g}}([\mathfrak{g}, \mathfrak{g}]) \cap [\mathfrak{g}, \mathfrak{g}]^\perp \neq \{0\}$, where $C_{\mathfrak{g}}([\mathfrak{g}, \mathfrak{g}])$ stands for the centralizer of $[\mathfrak{g}, \mathfrak{g}]$ in $\mathfrak{g}$.*

Then $\mathfrak{g}$ is a symplectic double extension of a symplectic Lie algebra by a line.

Proof. According to the proposition above it suffices to show the existence of a one-dimensional ideal orthogonal to $[\mathfrak{g}, \mathfrak{g}]$. If $\mathfrak{z}(\mathfrak{g}) \neq \{0\}$ then one may take an arbitrary central element $e \in \mathfrak{z}(\mathfrak{g})$ which obviously is orthogonal to $[\mathfrak{g}, \mathfrak{g}]$ because ω is a symplectic form. Suppose now that $\mathbb{K}$ is algebraically closed and $\mathfrak{L} = C_{\mathfrak{g}}([\mathfrak{g}, \mathfrak{g}]) \cap [\mathfrak{g}, \mathfrak{g}]^\perp \neq \{0\}$. Using that ω is a symplectic form, it is straightforward to show that $\mathfrak{L}$ is an ideal of $\mathfrak{g}$ and hence stable by the adjoint representation of $\mathfrak{g}$. Lie's theorem ensures then the existence of an element $e \in \mathfrak{L}$ and $\rho \in \mathfrak{g}^*$ such that $[x, e] = \rho(x)e$ for all $x \in \mathfrak{g}$ and the result follows. □

Remark 5.1. When $(\mathfrak{g}, \omega, K)$ is a para-Kähler Lie algebra with abelian paracomplex structure then either $\mathfrak{g}$ is abelian (and hence with non-trivial center) or $\{0\} \neq [\mathfrak{g}, \mathfrak{g}] \subset C_{\mathfrak{g}}([\mathfrak{g}, \mathfrak{g}]) \cap [\mathfrak{g}, \mathfrak{g}]^\perp$. Therefore, the corollary above ensures that if $\mathbb{K}$ is algebraically closed then $\mathfrak{g}$ is a symplectic double extension by a line of some symplectic Lie algebra $\mathfrak{h}$. However, in order to apply the result successively, one needs the Lie algebra $\mathfrak{g}$ to admit also an abelian para-Kähler structure. To guarantee such condition, it is convenient to ask the ideal $\mathbb{K}e$ to be K-invariant. This justifies the following definition:

Definition 5.2. We will say that a triple $(\tilde{\mathfrak{g}}, \tilde{\omega}, \tilde{K})$ is an *abelian para-Kähler double extension by a line* of a Lie algebra with abelian para-Kähler structure $(\mathfrak{g}, \omega, K)$ if the pair $(\tilde{\mathfrak{g}} = \mathbb{K}e \oplus \mathfrak{g} \oplus \mathbb{K}d, \tilde{\omega})$ is the symplectic double extension of $(\mathfrak{g}, \omega)$ by a line and $\tilde{K}$ is an abelian paracomplex structure on $\tilde{\mathfrak{g}}$ such that $\tilde{K}_{|\mathfrak{g}} = K$ and both elements e, d are eigenvectors of $\tilde{K}$ associated to opposite eigenvalues. Obviously, $(\tilde{\mathfrak{g}}, \tilde{\omega}, \tilde{K})$ is a para-Kähler Lie algebra.

Lemma 5.3. *A para-Kähler Lie algebra $(\tilde{\mathfrak{g}}, \tilde{\omega}, \tilde{K})$ with an abelian paracomplex structure is an abelian para-Kähler double extension of a symplectic Lie algebra by a line if and only if there exists $e \in [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]^\perp$ such that $\mathbb{K}e$ is an ideal of $\tilde{\mathfrak{g}}$ stable by $\tilde{K}$.*

Proof. Note that if W_1, W_2 denote the eigenspaces of $\tilde{K}$ and $e \in W_1$, then one may find $d \in W_2$ such that $\tilde{\omega}(d, e) = 1$. Now, the proof is just a simple consequence of Proposition 5.1 since the decomposition $\tilde{\mathfrak{g}} = \mathbb{K}e \oplus \mathfrak{g} \oplus \mathbb{K}d$ remains invariant under K. □

Proposition 5.4. *Let $(\tilde{\mathfrak{g}}, \tilde{\omega}, \tilde{K})$ be a para-Kähler Lie algebra with abelian paracomplex structure and suppose that* $\dim(\tilde{\mathfrak{g}}) > 2$.

(i) *If $\mathfrak{z}(\tilde{\mathfrak{g}}) \neq \{0\}$, then $\tilde{\mathfrak{g}}$ is an abelian para-Kähler double extension of a Lie algebra $(\mathfrak{g}, \omega, K)$ with an abelian para-Kähler structure.*
(ii) *In particular, if $\tilde{\mathfrak{g}}$ is nilpotent, then it can be obtained by a sequence of abelian para-Kähler double extensions by lines starting from the abelian 2-dimensional Lie algebra.*

Proof. In order to prove (i) we first show that one can always find $e \in \mathfrak{z}(\tilde{\mathfrak{g}})$ which is an eigenvector for $\tilde{K}$. Let W_1, W_2 be the eigenspaces of K and take a central element $z = w_1 + w_2 \neq 0$ where $w_1 \in W_1$, $w_2 \in W_2$. Since $[W_i, W_i] = \{0\}$ for $i = 1, 2$, one immediately obtains that $w_1, w_2 \in \mathfrak{z}(\tilde{\mathfrak{g}})$. As there exists $i \in \{1, 2\}$ such that $w_i \neq 0$, we can take $e = w_i$. Now, the proof follows at once from the lemma above.

When the Lie algebra $\tilde{\mathfrak{g}}$ is nilpotent, it has a non-trivial center and therefore it is an abelian para-Kähler double extension of a Lie algebra $\mathfrak{g}$ which also admits an abelian para-Kähler structure. Since $\mathfrak{g}$ must also be nilpotent, we can apply the same reasoning successively and the result follows. □

The next lemma shows that not every para-Kähler Lie algebra with abelian paracomplex structure is an abelian para-Kähler double extension.

Lemma 5.5. *Let $(\mathfrak{g}, \omega, K)$ be a Lie algebra with abelian para-Kähler structure and let W_1, W_2 be the eigenspaces of K. If $\mathfrak{g}$ admits a K-invariant one-dimensional ideal, then either $\operatorname{ann}(W_1) \neq \{0\}$ or $\operatorname{ann}(W_2) \neq \{0\}$, where $\operatorname{ann}(W_i)$ stands for the (two-sided) annihilator of the associative commutative algebra W_i.*

Proof. If $\mathbb{K}e$ is K-invariant, then $e \in W_1$ or $e \in W_2$. We can suppose without loss of generality that $e \in W_1$. Since $\mathbb{K}e$ is an ideal of $\mathfrak{g}$ there exists $\rho \in \mathfrak{g}^*$ such that $[e, x] = \rho(x)e$ for all $x \in \mathfrak{g}$. Therefore, for every $w_1 \in W_1, w_2 \in W_2$ we have

$$0 = \omega\big(\rho(w_2)e, w_1\big) = \omega\big([e, w_2], w_1\big) = -\omega(w_2, e.w_1),$$

which obviously implies $e \in \operatorname{ann}(W_1)$. □

In the following results we characterize those such algebras with abelian para-Kähler structure for which both annihilators of the eigenspaces are null.

Lemma 5.6. *Let us consider a para-Kähler Lie algebra with abelian paracomplex structure $(\mathfrak{g}, \omega, K)$ and consider the Lie subalgebra $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}] + K[\mathfrak{g}, \mathfrak{g}]$.*

If, for each $x \in \mathfrak{s}$, we denote by $\mathcal{R}_x : \mathfrak{g} \to \mathfrak{g}$ the right multiplication map $\mathcal{R}_x(a) = a.x$, $a \in \mathfrak{g}$, for the left-symmetric product defined by ω, then $\mathcal{R}_x \circ \mathcal{R}_y = \mathcal{R}_{y.x}$ for all $x, y \in \mathfrak{s}$. As a consequence, $(\mathfrak{s}, .)$ is associative.

Proof. Recall that if $U = \ker(K - \mathrm{id}_\mathfrak{g})$, $U^* = \ker(K + \mathrm{id}_\mathfrak{g})$, then $\mathfrak{s} = U.U^* + U^*.U$ and, therefore, it suffices to prove the identity for $x, y \in \mathfrak{s}$ of the form $u.f$ or $f.u$ where $u \in U$ and $f \in U^*$. Further, since $\mathcal{R}_{u_1} \circ \mathcal{R}_{u_2} = \mathcal{R}_{u_2.u_1}$ and $\mathcal{R}_{f_1} \circ \mathcal{R}_{f_2} = \mathcal{R}_{f_2.f_1}$ hold for all $u_1, u_2 \in U$, $f_1, f_2 \in U^*$, it will be sufficient to prove the equalities

$$\mathcal{R}_{u_1.f_1} \circ \mathcal{R}_{f_2.u_2} = \mathcal{R}_{(f_2.u_2).(u_1.f_1)}, \qquad \mathcal{R}_{f_1.u_1} \circ \mathcal{R}_{u_2.f_2} = \mathcal{R}_{(u_2.f_2).(f_1.u_1)}$$

for all $u_1, u_2 \in U$, $f_1, f_2 \in U^*$.

It is obvious that for $u_1, u_2 \in U$, $f_1, f_2 \in U^*$ one has $\mathcal{R}_{u_1.f_1} \circ \mathcal{R}_{f_2.u_2}(u) = \mathcal{R}_{(f_2.u_2).(u_1.f_1)}(u)$ for all $u \in U$ because the relation $\mathcal{R}_{\tilde{u}} \circ \mathcal{R}_{\tilde{f}}(u) = \mathcal{R}_{\tilde{f}.\tilde{u}}(u)$ holds for all $\tilde{u}, u \in U$, $\tilde{f} \in U^*$. Now, if we take $u \in U$, $f \in U^*$, we have:

$$\begin{aligned}
\big(\mathcal{R}_{u_1.f_1} \circ \mathcal{R}_{f_2.u_2}(f)\big)(u) &= \big(\big(f.(f_2.u_2)\big).(u_1.f_1)\big)(u) = \big(f.(f_2.u_2)\big)\big(u.(u_1.f_1)\big) \\
&= \big(\big(f.(f_2.u_2)\big).u\big)(u_1.f_1) = \big(\big(\big(f.(f_2.u_2)\big).u\big).f_1\big)(u_1) = \big(\big(f.(f_2.u_2)\big).(f_1.u)\big)(u_1) \\
&= \big(\big(f.(f_2.u_2)\big).(f_1.u_1)\big)(u) = \big(f.(f_2.u_2).(f_1.u_1)\big)(u)
\end{aligned}$$

and also

$$\begin{aligned}
\big(\mathcal{R}_{(f_2.u_2).(u_1.f_1)}(f)\big)(u) &= \big(f.\big((f_2.u_2).(u_1.f_1)\big)\big)(u) = \big(f.\big((f_2.u_2).u\big)\big)(u_1.f_1) \\
&= \big(\big(f.\big((f_2.u_2)\big).u\big).f_1\big)(u_1) = \big(\big(f.\big((f_2.u_2)\big).u_1\big).f_1\big)(u) = \big(f.(f_2.u_2).(f_1.u_1)\big)(u),
\end{aligned}$$

which proves that $\mathcal{R}_{u_1.f_1} \circ \mathcal{R}_{f_2.u_2}(f) = \mathcal{R}_{(f_2.u_2).(u_1.f_1)}(f)$.

On the other hand, the identity $\mathcal{R}_{\tilde{f}} \circ \mathcal{R}_{\tilde{u}}(f) = \mathcal{R}_{\tilde{u}.\tilde{f}}(f)$ is verified for all $\tilde{u} \in U$, $\tilde{f}, f \in U^*$ and if we take $u \in U$, then

$$\begin{aligned}
f\big(\mathcal{R}_{f_1.u_1} \circ \mathcal{R}_{u_2.f_2}(u)\big) &= f\big(\big(u.(u_2.f_2)\big).(f_1.u_1)\big) = \big(\big(f.(f_1.u_1)\big).u\big)(u_2.f_2) \\
&= \big(\big(\big(f.(f_1.u_1)\big).u\big).f_2\big)(u_2) = \big(\big(f.(f_1.u_1)\big).(f_2.u)\big)(u_2) = \big(\big(f.(f_1.u_1)\big).(f_2.u_2)\big)(u)
\end{aligned}$$

and, similarly,

$$\begin{aligned}
f\big(\mathcal{R}_{(u_2.f_2).(f_1.u_1)}(u)\big) &= f\big(u.\big((u_2.f_2).(f_1.u_1)\big)\big) = (f.u)\big((u_2.f_2).(f_1.u_1)\big) \\
&= \big((f.u).(f_1.u_1).f_2\big)(u_2) = \big((f.u_2).(f_1.u_1).f_2\big)(u) = \big(\big(f.(f_1.u_1)\big).(f_2.u_2)\big)(u).
\end{aligned}$$

This proves that $\mathcal{R}_{f_1.u_1} \circ \mathcal{R}_{u_2.f_2}(u) = \mathcal{R}_{(u_2.f_2).(f_1.u_1)}(u)$ for all $u \in U$.

Next, notice that $(\mathfrak{s}, .)$ is a subalgebra for the left-symmetric product on $\mathfrak{g}$ since the products $(f_1.u_1).(u_2.f_2)$ and $(u_1.f_1).(f_2.u_2)$ are obviously in $\mathfrak{s} = U.U^* + U^*.U$ and we also have

$$(u_1.f_1).(u_2.f_2) = u_1.\big(f_1.(u_2.f_2)\big) \in U.U^*,$$
$$(f_1.u_1).(f_2.u_2) = f_1.\big(u_1.(f_2.u_2)\big) \in U^*.U.$$

Now, the associativity of $(\mathfrak{s}, .)$ is a direct consequence of $\mathcal{R}_x \circ \mathcal{R}_y = \mathcal{R}_{y.x}$ for $x, y \in \mathfrak{s}$. $\square$

A word-by-word translation of the proof of Barberis and Dotti's Proposition 4.1 in [5] by simply changing the complex structure by the paracomplex structure provides the following:

Proposition 5.7. *Let $(\mathfrak{s}, K)$ be a solvable Lie algebra with an abelian paracomplex structure and $\mathfrak{z}(\mathfrak{s})$ its centre. If $\mathfrak{s}$ admits a decomposition $\mathfrak{s} = \mathfrak{u} + K\mathfrak{u}$ where $\mathfrak{u}$ is an abelian ideal, then $\mathfrak{s}/\mathfrak{z}(\mathfrak{s})$ is paraholomorphically isomorphic to $\mathfrak{aff}(A)$ for some commutative associative algebra A.*

However, for our purposes, it will be convenient a similar result for the quotient of the Lie subalgebra $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}] + K[\mathfrak{g}, \mathfrak{g}]$ by its ideal $\ker(\omega_\mathfrak{s}) = \mathfrak{s} \cap \mathfrak{s}^\perp$. Since $\mathfrak{s} = U.U^* + U^*.U$ one can easily prove that $\ker(\omega_\mathfrak{s}) \subset \mathfrak{z}(\mathfrak{s})$ because for every $x \in \ker(\omega_\mathfrak{s})$ and $u, u' \in U$, $f, f' \in U^*$ one has

$$\omega\big([x, u.f], u'\big) = \omega\big(x, (u.f).u'\big) = \omega\big(x, u.(f.u')\big) = 0, \qquad \omega\big([x, u.f], f'\big) = \omega\big(x, (u.f).f'\big) = 0$$

and analogous results can be obtained for $[x, f.u]$. Thus, our next proposition is, somehow, a stronger version of the one above.

Proposition 5.8. *Let $(\mathfrak{g}, \omega, K)$ be a para-Kähler Lie algebra with abelian paracomplex structure and $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}] + K[\mathfrak{g}, \mathfrak{g}]$. Let us denote by $\omega_\mathfrak{s}$ the restriction of ω to $\mathfrak{s} \times \mathfrak{s}$ and by $\ker(\omega_\mathfrak{s})$ its kernel.*

The quotient Lie algebra $\mathfrak{s}/\ker(\omega_\mathfrak{s})$ is isomorphic to $\mathfrak{aff}(M)$ where M is the subalgebra of $\mathfrak{gl}(\mathfrak{g})$ given by

$$M = \{\mathcal{R}_{Kx}\colon x \in [\mathfrak{g}, \mathfrak{g}]\}.$$

Further, the isomorphism is paraholomorphic if on $\mathfrak{aff}(M)$ one considers the paracomplex structure $K_\mathfrak{a}(a, b) = (b, a)$ for $a, b \in M$ and on $\mathfrak{s}/\ker(\omega_\mathfrak{s})$ the one induced by K.

Proof. Let first note that $[\mathfrak{g}, \mathfrak{g}] \cap K[\mathfrak{g}, \mathfrak{g}] \subset \ker(\omega_\mathfrak{s})$ since if $x = Kx'$, where $x, x' \in [\mathfrak{g}, \mathfrak{g}]$, and $h, h' \in [\mathfrak{g}, \mathfrak{g}]$ then

$$\omega\big(x, h + Kh'\big) = \omega(x, h) + \omega\big(x, Kh'\big) = \omega(x, h) + \omega\big(Kx', Kh'\big) = \omega(x, h) - \omega\big(x', h'\big)$$

and both summands in the right-hand expression vanish because $\omega([\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]) = \{0\}$.

Since $[Kx, Kx'] = -[x, x'] = 0$ for all $x, x' \in [\mathfrak{g}, \mathfrak{g}]$ because $\mathfrak{g}$ is 2-step solvable, it is obvious from Lemma 5.6 above that M is a commutative (and associative) algebra. Let us define a linear map $\psi : \mathfrak{s} \to \mathfrak{aff}(M)$ by

$$\psi(x + Ky) = (-\mathcal{R}_{Ky}, -\mathcal{R}_{Kx}), \quad x, y \in [\mathfrak{g}, \mathfrak{g}].$$

If $x, y \in [\mathfrak{g}, \mathfrak{g}]$, then $x + Ky \in \ker(\omega_\mathfrak{s})$ if and only if for every $x_1, y_1, x_2, y_2 \in \mathfrak{g}$,

$$\begin{aligned} 0 &= \omega\big(x + Ky, [x_1, y_1] + K[x_2, y_2]\big) = \omega\big(x, K[x_2, y_2]\big) + \omega\big(Ky, [x_1, y_1]\big) \\ &= \omega(x_2.Kx, y_2) - \omega(x_1.Ky, y_1) = \omega\big(\mathcal{R}_{Kx}(x_2), y_2\big) - \omega\big(\mathcal{R}_{Ky}(x_1), y_1\big). \end{aligned}$$

This implies that ψ is well-defined and that $\ker(\psi) = \ker(\omega_\mathfrak{s})$. Therefore, since ψ is obviously onto, it suffices to see that it is a Lie algebras homomorphism. Consider $x, y, x', y' \in [\mathfrak{g}, \mathfrak{g}]$. Recall that we had that $\mathfrak{g}.[\mathfrak{g}, \mathfrak{g}] = \{0\}$ and that K commutes with right multiplications and, thus,

$$K\big(\big[x + Ky, x' + Ky'\big]\big) = K\big[x, Ky'\big] + K\big[Ky, x'\big] = K\big(x.Ky'\big) - K\big(x'.Ky\big) = Kx.Ky' - Kx'.Ky.$$

Then, as $[x + Ky, x' + Ky'] \in [\mathfrak{g}, \mathfrak{g}]$, one gets using Lemma 5.6 that

$$\psi\big(\big[x + Ky, x' + Ky'\big]\big) = (0, -\mathcal{R}_{Kx.Ky' - Kx'.Ky}) = (0, -\mathcal{R}_{Ky'} \circ \mathcal{R}_{Kx} + \mathcal{R}_{Ky} \circ \mathcal{R}_{Kx'}).$$

But, on the other hand, we have

$$\big[\psi(x + Ky), \psi\big(x' + Ky'\big)\big] = \big[(-\mathcal{R}_{Ky}, -\mathcal{R}_{Kx}), (-\mathcal{R}_{Ky'}, -\mathcal{R}_{Kx'})\big] = (0, \mathcal{R}_{Ky} \circ \mathcal{R}_{Kx'} - \mathcal{R}_{Ky'} \circ \mathcal{R}_{Kx}),$$

which proves that ψ is a homomorphism.

Finally, for every $x, x' \in [\mathfrak{g}, \mathfrak{g}]$ we obtain

$$\psi \circ K(x + Kx') = \psi(x' + Kx) = (-\mathcal{R}_{Kx}, -\mathcal{R}_{Kx'})$$
$$= K_{\mathfrak{a}}(-\mathcal{R}_{Kx'}, -\mathcal{R}_{Kx}) = K_{\mathfrak{a}} \circ \psi(x + Kx')$$

and therefore, since K leaves the kernel of $\omega_{\mathfrak{s}}$ invariant, the isomorphism is actually paraholomorphic. □

Corollary 5.9. *Let $(\mathfrak{g}, \omega, K)$ be a para-Kähler Lie algebra with abelian paracomplex structure, $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}] + K[\mathfrak{g}, \mathfrak{g}]$ and let $\tilde{\omega}$ denote the symplectic structure induced by ω on the quotient $\mathfrak{s}/\ker(\omega_{\mathfrak{s}})$. Consider $\tilde{\psi} : \mathfrak{s}/\ker(\omega_{\mathfrak{s}}) \to \mathfrak{aff}(M)$ the isomorphism given by the proposition above.*

The symplectic structure on $\mathfrak{aff}(M)$ defined by $\omega_{\mathfrak{a}} = (\tilde{\psi}^{-1})^*\tilde{\omega}$ induces on $\mathfrak{aff}(M)$ the left symmetric (actually, associative) product

$$(a, b).(a', b') = (-aa', -ba'), \quad a, b, a', b' \in M.$$

The associative commutative algebra M is endowed with a non-degenerate symmetric bilinear form B such that $B(ab, c) = B(a, bc)$ holds for every $a, b, c \in M$.

Proof. Since $\tilde{\psi}$ is an isomorphism of Lie algebras and we have defined $\omega_{\mathfrak{a}} = (\tilde{\psi}^{-1})^*\tilde{\omega}$, then it is straightforward to see that $\tilde{\psi}$ is actually an isomorphism of the left-symmetric structures defined by $\tilde{\omega}$ and $\omega_{\mathfrak{a}}$. Therefore, if we set $a = \mathcal{R}_{Kx}$, $b = \mathcal{R}_{Ky}$ for $x, y \in [\mathfrak{g}, \mathfrak{g}]$ then we have

$$\begin{aligned}
(a, 0).(b, 0) &= \tilde{\psi}\big(\tilde{\psi}^{-1}(a, 0).\tilde{\psi}^{-1}(b, 0)\big) = \psi\big((-Kx).(-Ky)\big) = \psi\big(K[x, Ky]\big) \\
&= (-\mathcal{R}_{K[x,Ky]}, 0) = (-\mathcal{R}_{Kx.Ky}, 0) = (-ab, 0), \\
(a, 0).(0, b) &= \tilde{\psi}\big(\tilde{\psi}^{-1}(a, 0).\tilde{\psi}^{-1}(0, b)\big) = \psi\big((-Kx).(-y)\big) = \psi\big(K(x.y)\big) = 0, \\
(0, a).(0, b) &= \tilde{\psi}\big(\tilde{\psi}^{-1}(0, a).\tilde{\psi}^{-1}(0, b)\big) = \psi\big((-x).(-y)\big) = \psi(x.y) = 0, \\
(0, a).(b, 0) &= \tilde{\psi}\big(\tilde{\psi}^{-1}(0, a).\tilde{\psi}^{-1}(b, 0)\big) = \psi\big((-x).(-Ky)\big) = \psi\big([x, Ky]\big) \\
&= (0, -\mathcal{R}_{K[x,Ky]}) = (0, -\mathcal{R}_{Kx.Ky}) = (0, -ab)
\end{aligned}$$

and, hence, the symmetric product on $\mathfrak{aff}(M)$ is as claimed. Note that in an example above we had proved that it is actually associative.

Define on M the bilinear form $B(a, b) = \omega_{\mathfrak{a}}((a, 0), (0, b))$. Now, notice that if $x, y \in [\mathfrak{g}, \mathfrak{g}]$, then $\omega(x, y) = \omega(Kx, Ky) = 0$ and $\omega(Kx, y) = -\omega(y, Kx) = \omega(Ky, x)$ and, hence, for $a = \mathcal{R}_{Kx}$, $a' = \mathcal{R}_{Kx'}$, $b' = \mathcal{R}_{Ky}$ one has

$$\omega_{\mathfrak{a}}\big((a, 0), (a', b')\big) = \omega(-Kx, -y - Kx') = \omega(Kx, y) = \omega_{\mathfrak{a}}\big((a, 0), (0, b')\big) = B(a, b').$$

This shows that B is non-degenerate since $\omega_{\mathfrak{a}}$ is so. Further, since $\omega(Kx, y) = \omega(Ky, x)$ for all $x, y \in [\mathfrak{g}, \mathfrak{g}]$ one also gets that B is symmetric. Thus, it suffices to show the invariance property. Consider again $a = \mathcal{R}_{Kx}$, $a' = \mathcal{R}_{Kx'}$, $b = \mathcal{R}_{Ky}$; using that $d\omega = 0$ we get

$$\begin{aligned}
B(ab, a') &= \omega_{\mathfrak{a}}\big((ab, 0), (0, a')\big) = \omega(Kx.Ky, x') = -\omega(x.Ky, Kx') \\
&= \omega(Ky.Kx', x) = \mathcal{B}(ba', a) = \mathcal{B}(a, ba').
\end{aligned}$$

This completes the proof. □

We can now characterize all Lie algebras with an abelian para-Kähler structure whose eigenspaces have null annihilator.

Corollary 5.10. *Let $(\mathfrak{g}, \omega, K)$ be a para-Kähler Lie algebra and let $U = \ker(K - \mathrm{id}_{\mathfrak{g}})$, $V = \ker(K + \mathrm{id}_{\mathfrak{g}})$. If $\mathrm{ann}(U) = \mathrm{ann}(V) = \{0\}$ then $U.V = U$, $V.U = V$ and there exist a symmetric associative commutative algebra M such that $\mathfrak{g} = \mathfrak{aff}(M)$ endowed with the standard abelian para-Kähler structure.*

Proof. Let us see that if $\mathrm{ann}(V) = \{0\}$ then $U.V = U$. Take $u_0 + v_0 \in (U.V)^\perp$, where $u_0 \in U$, $v_0 \in V$ and consider $u \in U$, $v \in V$. We have

$$0 = \omega(u_0 + v_0, u.v) = \omega(v_0, [u, v]) = \omega(v.v_0, u)$$

which shows that $v_0 \in \mathrm{ann}(V) = \{0\}$. Hence, $(U.V)^\perp \subset U$. Since one trivially has $U \subset (U.V)^\perp$ because U is completely isotropic, we have that actually $U.V = U^\perp = U$. The analogous reasoning proves that $V.U = V$ whenever $\mathrm{ann}(U) = \{0\}$.

Now, according to Proposition 5.8 and its corollary, there exists a symmetric associative commutative algebra M such that $\mathfrak{s}/\ker(\omega_{\mathfrak{s}}) = \mathfrak{aff}(M)$ where $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}] + K[\mathfrak{g}, \mathfrak{g}]$. But we have that $\mathfrak{s} = U.V + V.U = U + V = \mathfrak{g}$ and since ω is non-degenerate, $\ker(\omega_{\mathfrak{s}}) = \ker(\omega) = \{0\}$ and, thus, $\mathfrak{g} = \mathfrak{aff}(M)$. □

Finally, the result above let us give a complete description of our Lie algebras in the case of an algebraically closed field:

Theorem 5.11. *If the field $\mathbb{K}$ is algebraically closed field, then every Lie algebra with an abelian para-Kähler structure either is an algebra $\mathfrak{aff}(M)$ for some symmetric associative commutative algebra M or can be obtained by a sequence of abelian para-Kähler double extensions by lines from an algebra $\mathfrak{aff}(M)$.*

Proof. Let $(\mathfrak{g}, \omega, K)$ be a para-Kähler Lie algebra with abelian K and let W_1, W_2 be the eigenspaces of K. If both eigenspaces have null annihilators then, by the corollary above, $\mathfrak{g}$ is $\mathfrak{aff}(M)$ for some M and the result follows. Thus, we may consider without loss of generality that $\operatorname{ann}(W_1) \neq \{0\}$. When $\mathfrak{z}(\mathfrak{g}) \neq \{0\}$ then we may apply Proposition 5.4(i) to see that $\mathfrak{g}$ is obtained by an abelian para-Kähler double extension. When $\mathfrak{g}$ is centerless, we consider $\mathcal{L} = \operatorname{ann}(W_1) \cap [\mathfrak{g}, \mathfrak{g}]$, which is an ideal of $\mathfrak{g}$ since by Corollary 4.5 $\operatorname{ann}(W_1)$ is an ideal of $\mathfrak{g}$. Further it cannot be null since $\mathfrak{z}(\mathfrak{g}) \neq \{0\}$ and thus, for every $w \in \operatorname{ann}(W_1)$ there exists $x \in \mathfrak{g}$ such that $0 \neq [w, x] \in \mathcal{L}$. Since $\mathbb{K}$ is algebraically closed, we can apply Lie's theorem to the representation of the solvable Lie algebra $\mathfrak{g}$ on $\mathcal{L}$ given by the adjoint representation and we get an element $e \in W_1$ such that $\mathbb{K}e$ is an ideal contained in $[\mathfrak{g}, \mathfrak{g}]^{\perp}$ and stable by the paracomplex structure K. By Lemma 5.3, $\mathfrak{g}$ is an abelian para-Kähler double extension by a line of another Lie algebra $\mathfrak{h}$ with an abelian para-Kähler structure.

Now, the stated result follows at once by applying the reasoning above to the algebra $\mathfrak{h}$ and repeating the process. $\square$

Remark 5.2. In principle, Theorem 5.11 is no longer valid if the base field is $\mathbb{R}$. However, given a real Lie algebra $(\mathfrak{g}, \omega, K)$ with an abelian para-Kähler structure, its complexification $\mathfrak{g}^{\mathbb{C}}$ may be naturally endowed with the symplectic and the abelian paracomplex structures defined by

$$\omega^{\mathbb{C}}\big(x + iy, x' + iy'\big) = \omega\big(x, x'\big) - \omega\big(y, y'\big) + i\big(\omega\big(x, y'\big) + \omega\big(y, x'\big)\big),$$
$$K^{\mathbb{C}}(x + iy) = K(x) + iK(y),$$

for all $x, x', y, y' \in \mathfrak{g}$. The triple $(\mathfrak{g}^{\mathbb{C}}, \omega^{\mathbb{C}}, K^{\mathbb{C}})$ results to be para-Kähler and, hence, the argument used in Theorem 5.11 is applicable to find an element $e = e_1 + ie_2$ such that $\mathbb{C}e$ is an ideal contained in $[\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}]^{\perp}$ which remains stable by $K^{\mathbb{C}}$. Actually, the linear span $\mathfrak{I}$ of $\{e_1, e_2\}$ is an (abelian) ideal of $\mathfrak{g}$, orthogonal to $[\mathfrak{g}, \mathfrak{g}]$ and stable by K. We can then find a vector subspace $\mathfrak{V} = \mathbb{R}\text{-span}\{d_1, d_2\} \subset \mathfrak{g}$ such that $[d_1, d_2] = 0$ and $\omega(d_i, e_j) = \delta_{ij}$. The Lie algebra $\mathfrak{g}$ is then obtained by a couple of extensions, a central extension first and a semidirect product afterwards, from the para-Kähler Lie algebra $\mathfrak{g} = \mathfrak{I}^{\perp}/\mathfrak{I}$ if we call this kind of extension $\mathfrak{g} = \mathfrak{I} \oplus \mathfrak{h} \oplus \mathfrak{V}$ an *abelian para-Kähler double extension by a plane* then one may prove the following:

Every real Lie algebra with an abelian para-Kähler structure either is an algebra $\mathfrak{aff}(M)$ for some symmetric associative commutative algebra M or can be obtained by a sequence of abelian para-Kähler double extensions by lines or planes from an algebra $\mathfrak{aff}(M)$.

6. Curvatures of the para-Kähler metric

Let $(\mathfrak{g}, \omega, K)$ be a para-Kähler Lie algebra with abelian paracomplex structure. On the Lie algebra $\mathfrak{g}$ we define the pseudo-Riemannian metric g defined by $g(x, y) = \omega(x, Ky)$. We will call g the *abelian para-Kähler metric* of $(\mathfrak{g}, \omega, K)$.

Proposition 6.1. *Let $(\mathfrak{g}, \omega, K)$ be a para-Kähler Lie algebra with abelian paracomplex structure and g the abelian para-Kähler metric. The Levi-Civita connection and the curvature tensor of g are respectively given for all $x, y, z \in \mathfrak{g}$ by*

$$\nabla_x y = Ky.Kx, \qquad R(x, y)z = Kz.K[x, y] - (z.Ky).Kx + (z.Kx).Ky.$$

Proof. The Levi-Civita connection may be computed by the Koszul formula which, for $x, y, z \in \mathfrak{g}$, reads

$$\begin{aligned}
2g(\nabla_x y, z) &= g\big([x, y], z\big) - g\big([y, z], x\big) + g\big([z, x], y\big)\\
&= \omega\big([x, y], Kz\big) - \omega\big([y, z], Kx\big) + \omega\big([z, x], Ky\big)\\
&= \omega\big([x, y], Kz\big) + \omega(z, y.Kx) + \omega(z, x.Ky)\\
&= \omega\big([x, y], Kz\big) + \omega\big(K(y.Kx), Kz\big) + \omega\big(K(x.Ky), Kz\big)\\
&= -\omega\big([Kx, Ky], Kz\big) + \omega\big(K(y.Kx), Kz\big) + \omega\big(K(x.Ky), Kz\big)\\
&= -g\big([Kx, Ky], z\big) + g\big(K(y.Kx), z\big) + g\big(K(x.Ky), z\big)
\end{aligned}$$

and therefore, since K commutes with right multiplications, we have

$$\nabla_x y = \frac{1}{2}(-Kx.Ky + Ky.Kx + Ky.Kx + Kx.Ky) = Ky.Kx,$$

for all $x, y \in \mathfrak{g}$.

Thus, for the curvature tensor, one has when $x, y, z \in \mathfrak{g}$

$$\begin{aligned} R(x, y)z &= \nabla_{[x,y]}z - \nabla_x\nabla_y z + \nabla_y\nabla_x z \\ &= Kz.K[x, y] - K(Kz.Ky).Kx + K(Kz.Kx).Ky \\ &= Kz.K[x, y] - (z.Ky).Kx + (z.Kx).Ky \end{aligned}$$

where, again, we have used the fact that K commutes with right multiplications. □

Remark 6.1. It should be noticed that the condition given by Lemma 2.1 is equivalent to the fact that K is parallel since

$$(\nabla_x K)(y) = \nabla_x Ky - K\nabla_x y = y.Kx - K(Ky.Kx) = 0$$

for every $x, y \in \mathfrak{g}$.

Corollary 6.2. *Let $(\mathfrak{g} = U \oplus U^*, \omega, K)$ be a para-Kähler Lie algebra with abelian paracomplex structure, where $U = \ker(K - \mathrm{id}_{\mathfrak{g}})$. The Riemannian curvature tensor for the para-Kähler metric is given by*

$$R\big(u, u'\big) = R\big(f, f'\big) = 0, \qquad R(u, f)f' = \big(f'.f\big)(u) - 3\big(f'.u\big).f, \qquad R(u, f)u' = 3\big(u'.f\big).u - \big(u'.u\big).f$$

for all $u, u' \in U$, $f, f' \in U^*$.

Proof. The proof of $R(u, u') = R(f, f') = 0$ follows for the associativity of the products on U and U^* and the properties (iii), (iv) of Proposition 4.4. Now, if we consider $u, u' \in U$, $f, f' \in U^*$ we have from Proposition 6.1 an Proposition 4.4:

$$\begin{aligned} R(u, f)u' &= u'.K[u, f] + \big(u'.f\big).u - \big(u'u\big).f = u'.(u.f) + u'.(f.u) + \big(u'.f\big).u - \big(u'u\big).f \\ &= 3\big(u'.f\big).u - \big(u'.u\big).f, \\ R(u, f)f' &= -f'.K[u, f] + \big(f'.f\big).u - \big(f'u\big).f = -f'.(u.f) - f'.(f.u) + \big(f'.f\big).u - \big(f'u\big).f \\ &= \big(f'.f\big).u - 3\big(f'.u\big).f, \end{aligned}$$

which finishes the proof. □

Proposition 6.3. *Let $(\mathfrak{g}, \omega, K)$ be a para-Kähler Lie algebra with abelian paracomplex structure and g the abelian para-Kähler metric.*

(i) *Suppose that the left-symmetric product induced by ω on $\mathfrak{g}$ is associative. The metric g is flat if and only if $\omega([\mathfrak{g}, \mathfrak{g}], K[\mathfrak{g}, \mathfrak{g}]) = \{0\}$.*
(ii) *As a consequence, if for some symmetric commutative associative algebra M one has that $\mathfrak{g} = \mathfrak{aff}(M)$ endowed with the standard para-Kähler structure, then the para-Kähler metric is flat if and only if $\mathfrak{g}$ is 2-step nilpotent.*
(iii) *If $\mathfrak{g}$ is 2-step nilpotent then g is flat if and only if the left-symmetric product induced by ω on $\mathfrak{g}$ is actually associative.*

Proof. If the left-symmetric product is associative, one has for every $x, y, z \in \mathfrak{g}$:

$$\begin{aligned} R(x, y)z &= Kz.K[x, y] - (z.Ky).Kx + (z.Kx).Ky = Kz.K[x, y] + z.[Kx, Ky] \\ &= Kz.K[x, y] - z.[x, y]. \end{aligned}$$

Therefore, it holds

$$\omega\big(R(x, y)z, t\big) = -\omega\big(K[x, y], [Kz, t]\big) + \omega\big([x, y], [z, t]\big) = -\omega\big(K[x, y], [Kz, t]\big).$$

for all $x, y, z, t \in \mathfrak{g}$, showing that the $R = 0$ if and only if $\omega([\mathfrak{g}, \mathfrak{g}], K[\mathfrak{g}, \mathfrak{g}]) = \{0\}$.

If $\mathfrak{g} = \mathfrak{aff}(M)$ endowed with the standard para-Kähler structure induced by a symmetric Frobenius form B on $\mathfrak{g}$, then the condition above implies that $M^3 = \{0\}$ since $[\mathfrak{g}, \mathfrak{g}] = \{0\} \times M^2$, $K[\mathfrak{g}, \mathfrak{g}] = M^2 \times \{0\}$ and, thus,

$$\{0\} = \omega\big([\mathfrak{g}, \mathfrak{g}], K[\mathfrak{g}, \mathfrak{g}]\big) = \omega\big(\{0\} \times M^2, M^2 \times \{0\}\big) = B\big(M^2, M^2\big) = B\big(M^3, M\big).$$

Since $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = \{0\} \times M^3$, the result follows.

Finally, when $\mathfrak{g}$ is 2-step nilpotent and $\mathfrak{g} = U \oplus U^*$ where $U = \ker(K - \mathrm{id}_{\mathfrak{g}})$ then we have

$$0 = \big[u', [f, u]\big] = u'.(f.u) - (f.u).u', \qquad 0 = \big[f', [f, u]\big] = (u.f).f' - f'.(u.f) \tag{2}$$

which obviously implies $(u'.f).u = u'.(f.u) = 0$ and $(f'.u).f = f'.(u.f) = 0$ for all $u, u' \in U$, $f, f' \in U^*$. Further, the condition $u'.(f.u) = 0$ means that $f'.(f.u) = 0$ because $f'.(f.u)(u') = f'(u'.(f.u))$. From the Corollary above, we then have that $R = 0$ if and only if $(f.f').u = 0$ and $(u'.u).f = 0$, but these lead us to

$$\left(f.f'\right)\circ R_u=\left(f.f'\right).u=0=f'.(f.u)$$

for all $u\in U$, $f,f'\in U^*$, which proves the associativity condition given in Remark 4.2. The converse follows immediately from (i) and (2) since $[\mathfrak{g},\mathfrak{g}]+K[\mathfrak{g},\mathfrak{g}]=U.U^*+U^*.U$. $\square$

Example 6.1. The result given in (ii) of the proposition above is no longer valid if on $\mathfrak{aff}(M)$ one considers a different abelian para-Kähler structure. For instance, on the non-nilpotent solvable Lie algebra $\mathfrak{aff}(\mathbb{R})$ let us consider a basis $\{x,y\}$ such that $[x,y]=y$ and take the symplectic form defined by $\omega(x,y)=1$ and the paracomplex structure $Kx=x$, $Ky=-y$. The left-symmetric product defined by ω is given by $yx=-y$, $x^2=-x$, $xy=y^2=0$, which is associative. Since the right multiplication by y is null, one easily sees that the metric is flat.

Let us denote by $\mathrm{Ric}(x,y)$ de Ricci curvature tensor of the metric, this is to say,

$$\mathrm{Ric}(x,y)=\mathrm{trace}\big(R(x,-)y\big),\quad x,y\in\mathfrak{g}.$$

We have the following:

Proposition 6.4. *Let $(\mathfrak{g}=U\oplus U^*,\omega,K)$ be a para-Kähler Lie algebra with abelian paracomplex structure, where $U=\ker(K-\mathrm{id}_{\mathfrak{g}})$, and denote by $\mathcal{L}_x$ the left multiplication by $x\in\mathfrak{g}$ for the left-symmetric product induced by ω.*

The Ricci curvature tensor for the para-Kähler metric is given by

$$\mathrm{Ric}\big(u,u'\big)=\mathrm{Ric}\big(f,f'\big)=0,\qquad \mathrm{Ric}(f,u)=-2\,\mathrm{trace}(\mathcal{L}_{u.f})=-2\,\mathrm{trace}(\mathcal{L}_{f.u})$$

for $u,u'\in U$, $f,f'\in U^*$.

Proof. Let us consider $u,u',u''\in U$, $f,f',f''\in U^*$. According to the corollary above, we have

$$R\big(u,u''\big)u'=0,\qquad R\big(u,f''\big)u'\in U,\qquad R\big(f,f''\big)f'=0,\qquad R\big(f,u''\big)f'\in U^*,$$

which clearly imply that $\mathrm{Ric}(u,u')=\mathrm{Ric}(f,f')=0$. Now, if we denote respectively left and right multiplications by $x\in\mathfrak{g}$ by $\mathcal{L}_x$ and $\mathcal{R}_x$ and by and $\mathcal{R}_{x|U}$ its restriction to U, using the properties of Proposition 4.4 one has for $u,u'\in U$, f, $f'\in U^*$ that $R(f,f')(u)=0$ and

$$\begin{aligned}R\big(f,u'\big)(u)&=-R\big(u',f\big)u=-3(u.f).u'+\big(u.u'\big).f=-2(u.f).u'-\big(u'.f\big).u+\big(u'.u\big).f\\&=-2\mathcal{L}_{u.f}\big(u'\big)-[\mathcal{R}_{f|U},\mathcal{R}_{u|U}]\big(u'\big).\end{aligned}$$

Moreover, since the restriction of $\mathcal{L}_{u.f}$ to U has the same trace as $\mathcal{L}_{u.f}$ because $\mathcal{L}_{u.f}(U^*)\subset U$, we get

$$\mathrm{Ric}(f,u)=-2\,\mathrm{trace}(\mathcal{L}_{u.f})+\mathrm{trace}\big([\mathcal{R}_{f|U},\mathcal{R}_{u|U}]\big)=-2\,\mathrm{trace}(\mathcal{L}_{u.f}).$$

The second equality follows at once since in a left-symmetric algebra $\mathcal{L}_{[x,y]}=[\mathcal{L}_x,\mathcal{L}_y]$ holds for all x,y. $\square$

As a consequence we obtain the following result. Note that a similar result in the case of (non-necessarily abelian) complex structures in the nilpotent case was proved in [9, Lemma 6.3].

Corollary 6.5. *Let $(\mathfrak{g},\omega,K)$ be a para-Kähler Lie algebra with abelian paracomplex structure. If $\mathfrak{g}$ is unimodular then the abelian para-Kähler metric is Ricci-flat. As a consequence, it is Ricci-flat whenever $\mathfrak{g}$ is nilpotent.*

Proof. The proof is straightforward bearing in mind that for the left-symmetric structure defined by ω one has $\mathcal{L}_x=-\mathrm{ad}(x)^*$ for every $x\in\mathfrak{g}$, where $\mathrm{ad}(x)^*$ denotes the dual map of $\mathrm{ad}(x)$ with respect to ω, and hence $\mathrm{trace}(\mathcal{L}_x)=-\mathrm{trace}(\mathrm{ad}^*(x))=-\mathrm{trace}(\mathrm{ad}(x))=0$. $\square$

In [1] homogeneous para-Kähler Einstein manifolds were studied. Our next result gives a complete characterization of the (non-Ricci-flat) Einstein case for our Lie algebras with abelian para-Kähler structure. Recall that the metric g is Einstein if there exists $\alpha\in\mathbb{K}$ such that $\mathrm{Ric}=\alpha g$.

Proposition 6.6. *If $(\mathfrak{g},\omega,K)$ is a Lie algebra with abelian para-Kähler structure and the para-Kähler metric is Einstein but not Ricci-flat then $\mathfrak{g}=\mathfrak{aff}(M)$, endowed with the standard abelian para-Kähler structure, where M is a semisimple associative commutative algebra and the symmetric Frobenius form on M is (up to a scalar) a trace form.*

Proof. Let us consider that $\mathrm{Ric} = \alpha g$ for $\alpha \neq 0$ and put $\mathfrak{g} = U \oplus U^*$, where $U = \ker(K - \mathrm{id}_{\mathfrak{g}})$. Suppose first that $\mathrm{ann}(U) \neq \{0\}$ and take a non-zero element $u_0 \in \mathrm{ann}(U)$. For every $f \in U^*$ and $u \in U$ we then have that $(u_0.f).u = (u.f).u_0 = 0$ and hence $\mathcal{L}_{u_0.f|U} = 0$. As $\mathcal{L}_{u_0.f}(U^*) \subset U$, we get:

$$\alpha g(f, u_0) = \mathrm{Ric}(f, u_0) = -2\,\mathrm{trace}(\mathcal{L}_{u_0.f}) = -2\,\mathrm{trace}(\mathcal{L}_{u_0.f|U}) = 0,$$

for all $f \in U^*$, a contradiction with the non-degeneracy of g. A similar argument may be done if $\mathrm{ann}(U^*) \neq \{0\}$ and, therefore, by Corollary 5.10 we have that $\mathfrak{g} = \mathfrak{aff}(M)$ for some symmetric associative commutative algebra of dimension n, endowed with the standard para-Kähler structure.

Recall that for $\mathfrak{aff}(M)$ with the standard structure one has $U = \{(a, a):\ a \in M\}$ and $U^* = V = \{(b, -b):\ b \in M\}$ and, thus, if $u = (a, a)$, $v = (b, -b)$ then we have $u.v = -(ab, ab)$. Further, for every $u' = (a', a') \in U$ one has $\mathcal{L}_{u.v}(u') = (aba', aba')$ and one can see that $\mathrm{trace}(\mathcal{L}_{u.v|U}) = \mathrm{trace}(L_a L_b)$ where L_s stands for the left (or right, since M is commutative) product by $s \in M$. Explicitly, if one takes any positive-definite bilinear form φ on M and a φ-orthonormal basis $\{e_i\}_{i=1}^n$ in M, then the bilinear form $\tilde{\varphi}((a, a), (b, b)) = \varphi(a, b)$ on U is positive-definite and admits the orthonormal basis $\{(e_i, e_i)_{i=1}^n\}$; we then have

$$\mathrm{trace}(\mathcal{L}_{u.v|U}) = \sum_{i=1}^{n} \tilde{\varphi}\big((abe_i, abe_i), (e_i, e_i)\big) = \sum_{i=1}^{n} \varphi(abe_i, e_i) = \mathrm{trace}(L_a L_b).$$

Since the left member is just $-\mathrm{Ric}(u, v)/2 = -\alpha g(u, v)/2$ we have that M admits a non-degenerate trace form and therefore it must be semisimple by [13, Theorem 4.5, p. 97]. Finally, if B denotes the symmetric Frobenius product on M, one has

$$-\frac{2}{\alpha}\,\mathrm{trace}(L_a L_b) = g\big((a, a), (b, -b)\big) = \omega\big((a, a), (-b, b)\big) = 2B(a, b),$$

that obviously implies $B(a, b) = -\frac{1}{\alpha}\,\mathrm{trace}(L_a L_b)$ for all $a, b \in M$. □

Remark 6.2. It should be recalled that the only semisimple associative commutative algebras over $\mathbb{C}$ are the direct sums of copies of $\mathbb{C}$, and over $\mathbb{R}$ are just the direct sums of different copies of $\mathbb{R}$ and $\mathbb{C}$.

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